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Algorithms for the γ-Algebra of Electromagnetic Form Factors*

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We consider the γ-algebra arising in the calculation of contributions from any allowable vertex part to the electromagnetic form factors. Simple algorithms are established which enable any form factor contribution to be written down as a function of Chebyshev polynomials whose argument is $P = 2 - q^2$, where q is the photon 4-momentum. These algorithms are particularly suitable for use in computer programmes for evaluating high-order vertex parts in perturbation theory.

1. INTRODUCTION

The size and speed of modern computers has made it possible to attempt to calculate sixth- and higher-order matrix elements defined by Feynman graphs [1]. It is essential that the processes of calculation be made as systematic as possible, so that rules of procedure may be fed into the computer. The three principal steps in carrying out these calculations are

(a) performing the momentum integrations,

(b) doing the γ-algebra,

(c) evaluating multiple integrals over the Feynman parameters.

A systematic procedure for performing the momentum integrations in any Feynman graphs was described in 1952 [2]; as a result, it is possible to formulate a set of rules for writing down any Feynman integrand as a function of the Feynman parameters, external momenta, and of γ-matrices, given the topology of the graph. It is not necessary to use these rules in fourth-order calculations, which are frequently done using dispersion techniques. For higher-order calculations, the generality and orderliness of the method makes it very suitable for use with a computer.

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The problem of evaluating Feynman parameter integrals over many variables is difficult, but adaptive routines have been developed to deal with this problem [3]. There still remain the problems of achieving sufficient accuracy and of dealing with singularities. Improved integration techniques are being investigated by the author and his collaborators [4], and it is intended to apply these techniques to Feynman variable integrations.

Techniques of γ-algebra have been studied by a number of investigators, and these techniques are useful in all types of field theoretic calculations, including Feynman graph calculations. Various computer programmes have been written to enable γ-algebra to be carried out automatically [5]; these programmes depend to some extent on algorithms which are incorporated in the programmes. Certain of these algorithms [6] deal with the problem of eliminating relativistic scalar products $\gamma_\rho \cdots \gamma^\rho$ ($\rho = 0, 1, 2, 3$) from matrix elements, a problem which arises quite generally. Other algorithms may be available for specific problems or classes of problem; in a previous paper [7], simple algorithms were established for the evaluation of contributions to the magnetic moment from any possible type of matrix element arising in any electromagnetic vertex part. Now that the evaluation of the sixth-order contribution to the electron moment is desirable and also within the bounds of possibility, these specialised algorithms are useful. It has proved possible, however, to generalise these algorithms to calculate contributions to the form factors for electromagnetic vertex parts in which the photon is off the mass shell. This paper establishes these algorithms, enabling all the γ-algebra for any electromagnetic form factor to be performed almost instantaneously.

1. As in the previous paper [7], referred to in future as *I*, we are studying the γ-algebra arising from Feynman graphs of the type shown in Fig. 1; the external

DIAGRAMS

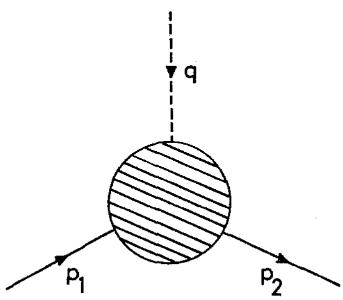

Figure 1

fermion line momenta are p_1 and p_2, and $q = p_2 - p_1$ is the photon line momentum. If p_1 and p_2 are free fermion momenta, the complete vertex part is of the form [8]

$$\bar{u}_2[F(q^2)\,\gamma_\mu + iG(q^2)\,\sigma_{\mu\nu}q^\nu]\,u_1\,, \tag{1.1}$$

even if $q^2 \neq 0$. The fermion mass is taken to be

$$M = 1, \tag{1.2}$$

so that p_1 and p_2 obey the free fermion equations

$$\bar{u}_2(\not{p}_2 - 1) = 0 \tag{1.3a}$$

and

$$(\not{p}_1 - 1)\,u_1 = 0, \tag{1.3b}$$

where $\not{p} = \gamma_\rho p^\rho$. In calculations, terms containing $p_{1\mu}$ and $p_{2\mu}$ arise symmetrically in the form

$$\bar{u}_2(p_1 + p_2)_\mu\,u_1 \equiv \bar{u}_2(2\gamma_\mu - i\sigma_{\mu\nu}\,q^\nu)\,u_1\,,$$

for any q; we have used Eqs (1.3). Thus we have, as in *I*, the first algorithm:

RULE 1.

$$\begin{aligned}\bar{u}_2\,p_{1\mu}\,u_1 &= \bar{u}_2\,p_{2\mu}\,u_1\\ &= \bar{u}_2(\gamma_\mu - \tfrac{1}{2}i\sigma_{\mu\nu}\,q^\nu)\,u_2\,.\end{aligned}$$

Equations (1.3) imply

$$p_1^2 = p_2^2 = 1, \tag{1.4}$$

but $p_1 \cdot p_2$ is not in general unity, as in I. We find

$$\begin{aligned}P \equiv 2p_1 \cdot p_2 &= p_1^2 + p_2^2 - q^2\\ &= 2 - q^2.\end{aligned} \tag{1.5}$$

The algorithms derived for calculating form factor contributions will be derived in two steps. First, we shall derive formulae for matrix elements which contain strings of $\not{p}_1$ and $\not{p}_2$ terms in which these terms alternate. The four possible types of strings are

$$E_1 = \not{p}_1\not{p}_2\not{p}_1\not{p}_2\cdots\not{p}_1\not{p}_2 \tag{1.6a}$$

$$E_2 = \not{p}_1\not{p}_2\not{p}_1\not{p}_2\cdots\not{p}_2\not{p}_1 \tag{1.6b}$$

$$E_3 = \not{p}_2\not{p}_1\not{p}_2\not{p}_1\cdots\not{p}_1\not{p}_2 \tag{1.6c}$$

$$E_4 = \not{p}_2\not{p}_1\not{p}_2\not{p}_1\cdots\not{p}_2\not{p}_1\,. \tag{1.6d}$$

More general elements, containing strings of $\not{p}_1$ and $\not{p}_2$ in any order, are reduced by reducing each string to one of the forms (1.6), using only the Klein–Gordon Eq. (1.4). We now establish an algorithm for reducing any string of $\not{p}_1$ and $\not{p}_2$ to one of the forms (1.6), and identifying the number of $\not{p}_1$ and $\not{p}_2$ in the reduced string E.

Consider, for example, a string

$$A = \overset{+}{\not{p}}_a \overset{-}{\not{p}}_b \overset{+}{\not{p}}_c \cdots \overset{+}{\not{p}}_e \overset{-}{\not{p}}_f \qquad (a, b, ..., e, f = 1, 2). \tag{1.7}$$

which reduces to the form E_1 by using (1.4) only; it must contain an even number of $\not{p}$, since E_1 does. We have labelled the string A with alternating $+$ and $-$ signs, starting at the left with a $+$ sign (we could, if we wished, start labelling at the right). Since A can be reduced to E_1, we are able to write A in the form

$$A = \overset{+-}{S_1} \overset{+}{\not{p}}_1 \overset{-+-}{S_2} \overset{}{\not{p}}_2 \overset{+-}{S_3} \overset{+}{\not{p}}_1 \cdots \overset{+}{\not{p}}_1 \overset{-+}{S_{k-1}} \overset{-}{\not{p}}_2 \overset{+-}{S_k}, \tag{1.8}$$

where $S_1, S_2, ..., S_k$ are strings of even numbers of $\not{p}$ which each reduce to the unit matrix 1 by using (1.4). In *I*, Eq. (3.7), it was shown that strings S which "reduce to unity" are identical with those in which

$$\begin{aligned} \text{number of } \overset{+}{\not{p}}_1 &= \text{number of } \overset{-}{\not{p}}_1 \\ \text{number of } \overset{+}{\not{p}}_2 &= \text{number of } \overset{-}{\not{p}}_2 \end{aligned} \tag{1.9}$$

Let us define, for a string A,

$$\alpha = (\text{number of } \overset{+}{\not{p}}_1) - (\text{number of } \overset{-}{\not{p}}_1) \tag{1.10}$$

and

$$\beta = (\text{number of } \overset{+}{\not{p}}_2) - (\text{number of } \overset{-}{\not{p}}_2). \tag{1.11}$$

Then the rule (1.9) tells us that omitting the strings $S_1, S_2, ..., S_k$ from (1.8) does not change α and β. So E_1 has the same values of α and β as (1.8). The same argument clearly also applies to strings A reducing to E_2, E_3 or E_4.

For a string E_1, and hence for a string A reducing to it,

$$\alpha = -\beta \geqslant 0.$$

The string E_1 can be immediately identified by using (1.10) and (1.11) to give α and β; we then know

(i) the string E_1 contains α terms $\overset{+}{\not{p}}_1$ and $|\beta| = \alpha$ terms $\overset{-}{\not{p}}_2$; and

(ii) that $\alpha > 0$ corresponds to the fact that E_1 has $\overset{+}{\not{p}}_1$ at the extreme left. (If $\alpha = \beta = 0$, the string just reduces to 1.

We can carry out a similar analysis of strings reducing to E_2, E_3 and E_4. Rule 2 below tells us how to identify any reduced string E, given α and β. We note that in any string

$$\alpha + \beta = 0 \text{ or } 1, \tag{1.12}$$

so that either

$$\alpha \geqslant 0 \qquad \text{or} \quad \beta \geqslant 0, \tag{1.13}$$

and

$$\alpha \leqslant 0 \qquad \text{or} \quad \beta \leqslant 0. \tag{1.14}$$

RULE 2.

(i) In any string A containing any number of $\not{p}_1$ and $\not{p}_2$ in any order, label the terms alternately with $+$ and $-$ signs, beginning at the left with a $+$ sign.

(ii) Calculate α and β for the string from (1.10) and (1.11).

(iii) If $\beta \leqslant 0$ (se that $\alpha \geqslant \beta$), the reduced alternating string E

(a) contains α of $\overset{+}{\not{p}}_1$ and $|\beta|$ of $\overset{-}{\not{p}}_2$,

(b) begins with $\overset{+}{\not{p}}_1$ on the left.

If $\alpha \leqslant 0$ (so that $\beta \geqslant \alpha$), the reduced alternating string E

(a) contains β of $\overset{+}{\not{p}}_2$ and $|\alpha|$ of $\overset{-}{\not{p}}_1$,

(b) begins with $\overset{+}{\not{p}}_2$ on the left.

It is clear that an alternative sign labelling, with the sign of the right-hand $\not{p}$ fixed, would give an equivalent rule.

2. MATRIX ELEMENTS NOT CONTAINING γ_μ

As in I, elimination of the scalar products $\gamma_\rho \cdots \gamma^\rho$ by known formulae [6] leads to vertex part matrix elements of three types:

$$(\mathrm{M}_1) \qquad (p_{1\mu}, p_{2\mu}) \cdot \bar{u}_2 \prod_a \not{p}_a u_1 \cdot \prod \mathrm{Tr}[\prod_b \not{p}_b];$$

$$(\mathrm{M}_2) \qquad \mathrm{Tr}[\gamma_\mu \prod_a \not{p}_a] \cdot \bar{u}_2 \prod_b \not{p}_b u_1 \cdot \prod \mathrm{Tr}[\prod_c \not{p}_c];$$

$$(\mathrm{M}_3) \qquad \bar{u}_2 \prod_a \not{p}_a \gamma_\mu \prod_b \not{p}_b \, u_1 \cdot \prod \mathrm{Tr}[\prod_c \not{p}_c].$$

In (M_1), (M_2), and (M_3), $\Pi_a \not{p}_a$ means that a product of several $\not{p}_1$ and $\not{p}_2$ may occur in any order; Π Tr means that several traces of this type may occur.

Factors $p_{1\mu}$ and $p_{2\mu}$ in (M_1) are dealt with by Rule 1. We consider next a factor of type

$$\bar{u}_2 \prod_a \not{p}_a u_1 \equiv \bar{u}_2 A u_1 . \tag{2.1}$$

Using (1.3) and (1.4), this can be reduced to the form

$$\Sigma_k \equiv \bar{u}_2 \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 \, u_1 , \tag{2.2}$$

with $2k$ terms in the string, $\not{p}_1$ and $\not{p}_2$ alternating, and $\not{p}_1$ on the left.

Now

$$\begin{aligned} \not{p}_1 \not{p}_2 &= 2p_1 \cdot p_2 - \not{p}_2 \not{p}_1 \\ &= P - \not{p}_2 \not{p}_1 \end{aligned}$$

so that for $k \geqslant 2$,

$$\Sigma_k(P) = \bar{u}_2(P - \not{p}_2 \not{p}_1) \overbrace{\not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2}^{2(k-1) \text{ terms}} . \tag{2.3}$$

$$= P\, \Sigma_{k-1}(P) - \Sigma_{k-2}(P). \tag{2.4}$$

Also

$$\Sigma_0(P) = \bar{u}_2 u_1 \tag{2.5}$$

and

$$\Sigma_1(P) = (P-1)\, \Sigma_0 . \tag{2.6}$$

Now the Chebyshev polynomials $S_k(P)$ and $C_k(P)$ obey the recurrence relation (2.4) and are of degree k in P. It is clear that $\Sigma_k(P)$ is also of degree k, so Σ_k is of the form

$$\Sigma_k(P) = AS_k(P) + BC_k(P).$$

Since

$$S_0(P) = 1, \qquad C_0(P) = 2,$$

and

$$S_1(P) = C_1(P) = P,$$

we can fix A and B, using (2.6). This gives

$$\Sigma_k(P) = [(1 - 2P^{-1})\, S_k(P) + P^{-1} C_k(P)]\, \Sigma_0 \, . \tag{2.7}$$

Using the relation

$$P^{-1}[C_k(P) - 2S_k(P)] = S_{k-1}(P), \tag{2.8}$$

(2.7) reduces to

$$\Sigma_k(P) = [S_k(P) - S_{k-1}(P)]\, \Sigma_0 \, . \tag{2.9}$$

We note that (2.9) holds for $k = 1$ and for $k = 0$, provided we define $S_{-1}(P) \equiv 0$.

When $q = 0$, $P = 2p_1 \cdot p_2 = 2$; then (2.9) gives

$$\begin{aligned} \Sigma_k(2) &= [S_k(2) - S_{k-1}(2)]\, \bar{u}_2 u_1 \\ &= \bar{u}_2 u_1 \, , \end{aligned}$$

which agrees with Rule 1 of I.

We therefore have a formula (2.9) for any matrix element of form (2.2), defined by the integer k. Given any matrix element (2.1), we need only to be able to identify k, which is given by Rule 2. Any string A reduces to one of the forms $E_1, \ldots, E_4$, defined by α and β; so the element (2.1) reduces to one of the forms

$$\bar{u}_2 E_s u_1 \qquad (s = 1, 2, 3, 4). \tag{2.10}$$

Use of the Dirac equations (1.3) reduces each of these four forms to the form Σ_k. We need to identify k in terms of α and β. Consider the four cases separately:

$\bar{u}_2 E_1 u_1$ ($\beta \leqslant 0$, $\alpha = -\beta$).

Dirac equation not used. Σ_k contains α of $\overset{+}{\not{p}}_1$, $|\beta| = \alpha$ of $\overset{-}{\not{p}}_2$.

$$k = \alpha = |\beta|$$

$\bar{u}_2 E_2 u_1$ ($\beta \leqslant 0$, $\alpha = 1 - \beta$).

Dirac equation (1.3b) used. Σ_k contains $(\alpha - 1)$ of $\overset{+}{\not{p}}_1$, $|\beta|$ of $\overset{-}{\not{p}}_2$.

$$k = \alpha - 1 = |\beta|$$

$\bar{u}_2 E_3 u_1$ ($\beta > 0$, $\alpha = 1 - \beta$).

Dirac equation (1.3a) used. Σ_k contains $(\beta - 1)$ of $\overset{+}{\not{p}}_2$, $|\alpha|$ of $\overset{-}{\not{p}}_1$.

$$k = \beta - 1 = |\alpha|$$

$\bar{u}_2 E_4 u_1$ $(\beta > 0, \alpha = -\beta)$.

Both Dirac equations (1.3) used. Σ_k contains $(\beta - 1)$ of $\overset{+}{\not{p}}_2$, $| \alpha - 1 |$ of $\overset{-}{\not{p}}_1$.

$$k = \beta - 1 = | \alpha - 1 | .$$

We can summarise these results in the form

$$\begin{aligned} &\text{if} \quad \beta \leqslant 0, k = | \beta | \\ &\text{if} \quad \beta > 0, k = \beta - 1 \end{aligned} \tag{2.11}$$

Thus we have the algorithm for calculating elements of type (2.1):

RULE 3.

(i) In any matrix element of the form

$$\bar{u}_2 \prod_a \not{p}_a u_1 \equiv \bar{u}_2 A u_1 ,$$

label the $\not{p}_a$ in A with signs as in Rule 1 and calculate β.

(ii) Define k by (2.11) in terms of β.

(iii) The matrix element then equals $\Sigma_k(P)$, given by (2.9) and (2.5), with $P = 2 - q^2$.

When $q = 0$ and hence $P = 2$, we should have $\Sigma_k = \Sigma_0$ for all k. Since $S_j(2) = j + 1$, equation (2.9) gives this result correctly.

3. TRACES

The arguments of Section 2 are easily adapted to calculate traces that may occur after the elimination of scalar products $\gamma_\alpha \cdots \gamma^\alpha$. These are of the form

$$\mathrm{Tr}[\prod_a \not{p}_a] \tag{3.1}$$

and

$$\mathrm{Tr}[\gamma_\mu \prod_a \not{p}_b]. \tag{3.2}$$

A trace of type (3.1) is zero if it contains an odd number of $\not{p}$. Otherwise it can be reduced by using (1.4) to one of the forms

$$\mathrm{Tr}[\not{p}_1 \not{p}_2 \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2] \tag{3.3a}$$

or

$$\mathrm{Tr}[\not{p}_2 \not{p}_1 \not{p}_2 \not{p}_1 \cdots \not{p}_2 \not{p}_1].$$

If we label the $\not{p}_a$ in (3.1) alternately with + and − signs, starting at the left with a + sign, and define α and β as in Rule 1, there will be

$$k = |\alpha| = |\beta| \tag{3.4}$$

pairs $(\not{p}_1 \not{p}_2)$ or $(\not{p}_2 \not{p}_1)$ in (3.3). Define (3.3a), for example, as

$$\Pi_k(P) = \mathrm{Tr}[\not{p}_1 \not{p}_2 \not{p}_1 \not{p}_2 \cdots (k \text{ pairs}) \cdots \not{p}_1 \not{p}_2]. \tag{3.5}$$

Then using (2.3), it follows that

$$\Pi_k(P) = P\Pi_{k-1}(P) - \Pi_{k-2}(P),$$

as for $\Sigma_k(P)$. Thus Π_k is of the form

$$\Pi_k(P) = XS_k(P) + YC_k(P). \tag{3.6}$$

Now $\Pi_0(P) = \mathrm{Tr}[1] = 4$

and $\Pi_1(P) = \mathrm{Tr}[p_1\, p_2]$

$$= \tfrac{1}{2}\,\mathrm{Tr}[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu]\, p_{1\nu}\, p_{2\nu}$$

$$= 4p_1 \cdot p_2 = 2P.$$

Putting these values into (3.6) when $k = 0, 1$, we find that $X = 0$ and $Y = 2$. Hence

$$\Pi_k(P) = 2C_k(P). \tag{3.7}$$

The algorithm for calculating (3.1) is thus:

RULE 4.

(i) Assume that the trace

$$\mathrm{Tr}[\prod_a \not{p}_a]$$

contains an even number of p_a, since it is otherwise zero.

(ii) Label the terms $\not{p}_a$ with + and − signs as in Rule 2, and find α or β.

(iii) Define $k = |\alpha| = |\beta|$. Then the trace is equal to $2C_k(P)$.

When $q = 0$ and $P = 2$, (3.7) gives

$$2C_k(2) = 4,$$

which is correct.

Traces of type (3.2), with an even number of $\not{p}_b$, are zero. With an odd number of $\not{p}_b$ factors they reduce, using (1.4), to either

$$\Xi_k = \mathrm{Tr}[\gamma_\mu \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 \not{p}_1] \tag{3.8a}$$

or to

$$\mathrm{Tr}[\gamma_\mu \not{p}_2 \not{p}_1 \cdots \not{p}_2 \not{p}_1 \not{p}_2], \tag{3.8b}$$

in which there are $(2k + 1)$ factors $\not{p}_1$ and $\not{p}_2$. We label the $\not{p}_b$ factors in $\Pi_b \not{p}_b$ in (3.2) with alternate signs, as in Rule 2, and define α and β satisfying

$$\alpha + \beta = 1.$$

Then k, defining the number of factors in (3.8), is given by Rule 2 as

$$k = \mathrm{Min}[|\alpha|, |\beta|]. \tag{3.9}$$

Now consider (3.8a) for example. Again using (2.3),

$$\Xi_k = P\Xi_{k-1} - \Xi_{k-2} \tag{3.10}$$

for $k \geqslant 2$, so that Ξ_k is again a linear combination of $S_k(P)$ and $C_k(P)$. Remembering that $p_{1\mu}$ and $p_{2\mu}$ give the same contribution p_μ, say, by Rule 1, we find

$$\Xi_0 = \mathrm{Tr}[\gamma_\mu \not{p}_1] = 4p_{1\mu} \to 4p_\mu$$

and

$$\begin{aligned} \Xi_1 &= \mathrm{Tr}[\gamma_\mu \not{p}_1 \not{p}_2 \not{p}_1] = \mathrm{Tr}[\gamma_\mu (P - \not{p}_2 \not{p}_1) \not{p}_1] \\ &= 4Pp_{1\mu} - 4p_{2\mu} \to 4(P-1)\, p_\mu . \end{aligned}$$

These values for Ξ_0 and Ξ_1 are the same as those for Σ_0 and Σ_1, given by (2.5) and (2.6), but with $\bar{u}_2 u_1$ replaced by $4p_\mu$. Thus for $k \geqslant 2$

$$\Xi_k(P) \to [S_k(P) - S_{k-1}(P)]\, 4p_\mu , \tag{3.11}$$

the analagous formula to (2.9). The factor p_μ has contributions given by Rule 1. It is clear that (3.9) and (3.11) apply equally to (3.8b). Also, if we define $S_{-1}(P) \equiv 0$, (3.11) applies for all $k \geqslant 0$.

When $q = 0$ and $P = 2$, (3.11) reduces to

$$\Xi_k(2) \to 4p_\mu ,$$

which is correct.

The algorithm for evaluating (3.2) is thus:

RULE 5.

(i) Assume that the trace

$$\mathrm{Tr}[\gamma_\mu \prod_b \not{p}_b]$$

contains an odd number of $\not{p}_b$, since it is otherwise zero.

(ii) Label the terms in $\Pi_b \not{p}_b$ with signs as in Rule 1, and calculate α and β for the string.

(iii) Define $k = \mathrm{Min}[|\alpha|, |\beta|]$. Then the trace reduces to

$$[S_k(P) - S_{k-1}(P)]\, 4p_\mu ,$$

where the contribution of $4p_\mu$ is given by Rule 1, and $S_{-1}(P) \equiv 0$.

4. MATRIX ELEMENTS CONTAINING γ_μ

The algorithms of Sections 2 and 3 deal with all traces and matrix elements that can occur, except for those of the form

$$\bar{u}_2 \prod_a \not{p}_a \gamma_\mu \prod_b \not{p}_b u_1 . \tag{4.1}$$

These are more complicated than previous types of term, since two strings of the form $\Pi_a \not{p}_a$ are involved. However, Rule 2 can be invoked to reduce each string to one of the forms $E_1 ,..., E_4$, given by (1.6). Use of the Dirac equations (1.3) gives a further reduction to the form

$$F \equiv \bar{u}_2 \not{p}_1 \not{p}_2 \not{p}_1 \cdots (m \text{ terms}) \cdots \gamma_\mu \cdots (n \text{ terms}) \cdots \not{p}_2 \not{p}_1 \not{p}_2\, u_1 . \tag{4.2}$$

We shall first of all evaluate the element (4.2); later we shall use Rule 2 to define m and n from the matrix element (4.1).

Consider an alternating string (of form E_1)

$$Q_k = \not{p}_1 \not{p}_2\, \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 , \tag{4.3}$$

containing $k(\geqslant 2)$ pairs $\not{p}_1 \not{p}_2$. Then

$$\begin{aligned} Q_k &= (P - \not{p}_2 \not{p}_1)\, \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 \\ &= P Q_{k-1} - Q_{k-2} . \end{aligned} \tag{4.4}$$

Once again, it follows that Q_k is of the form

$$Q_k = HS_k(P) + KC_k(P).$$

Putting $Q_0 = 1$ and $Q_1 = \not{p}_1 \not{p}_2$ fixes H and K, giving

$$Q_k = 1S_k(P) - \not{p}_2 \not{p}_1 S_{k-1}(P). \tag{4.5}$$

All elements of the form (4.2) can be written in one of four ways, corresponding to the choices of m, n odd or even:

$$F_1 = \bar{u}_2 Q_k \gamma_\mu Q_l u_1 \quad (m = 2k, n = 2l), \tag{4.6a}$$

$$F_2 = \bar{u}_2 Q_k \not{p}_1 \gamma_\mu Q_l u_1 \quad (m = 2k + 1, n = 2l), \tag{4.6b}$$

$$F_3 = \bar{u}_2 Q_k \gamma_\mu \not{p}_2 Q_l u_1 \quad (m = 2k, n = 2l + 1), \tag{4.6c}$$

$$F_4 = \bar{u}_2 Q_k \not{p}_1 \gamma_\mu \not{p}_2 Q_l u_1 \quad (m = 2k + 1, n = 2l + 1). \tag{4.6d}$$

Substituting (4.5) into Eqs. (4.6) and using (1.3) is equivalent to substituting

$$Q_k = 1S_k - \not{p}_1 S_{k-1} \tag{4.7a}$$

and

$$Q_l = 1S_l - \not{p}_2 S_{l-1} \tag{4.7b}$$

in (4.6). (The argument P of the Chebyshev functions is omitted.) All elements (4.6) then reduce to linear combinations of the following matrix elements:

$$\bar{u}_2 \gamma_\mu u_1 ,$$

$$\bar{u}_2 \not{p}_1 \gamma_\mu u_1 = \bar{u}_2 (2p_{1\mu} - \gamma_\mu)\, u_1 , \tag{4.8a}$$

$$\bar{u}_2 \gamma_\mu \not{p}_2 u_1 = \bar{u}_2 (2p_{2\mu} - \gamma_\mu)\, u_1 , \tag{4.8b}$$

$$\bar{u}_2 \not{p}_1 \gamma_\mu \not{p}_2 u_1 = \bar{u}_2 [2(p_{1\mu} + p_{2\mu}) - (P + 1)\, \gamma_\mu]\, u_1 . \tag{4.8c}$$

Remembering that $p_{1\mu}$ and $p_{2\mu}$ give equal contributions (p_μ) by Rule 1, we use (4.7) and (4.8) to express the matrix parts of F_1 ,..., F_4 in terms of γ_μ and p_μ as follows.

$(m = 2k, n = 2l)$:

$$\begin{aligned} F_1 &= \bar{u}_2 (S_k - \not{p}_1 S_{k-1})\, \gamma_\mu (S_l - \not{p}_2 S_{l-1})\, u_1 , \\ &\to S_k S_l \gamma_\mu - (S_{k-1} S_l + S_k S_{l-1})(2p_\mu - \gamma_\mu) \\ &\quad + S_{k-1} S_{l-1} [4p_\mu - (P + 1)\, \gamma_\mu]. \end{aligned}$$

The coefficients of γ_μ and of p_μ are

$$\gamma_\mu : (S_k + S_{k-1})(S_l + S_{l-1}) - (P+2)\, S_{k-1}S_{l-1}\,, \tag{4.9a}$$

$$p_\mu : -2[S_{k-1}(S_l - S_{l-1}) + S_{l-1}(S_k - S_{k-1})]. \tag{4.10a}$$

$(m = 2k+1, n = 2l)$:

$$\begin{aligned} F_2 &= \bar{u}_2(S_k - \not{p}_1 S_{k-1})\, \not{p}_1 \gamma_\mu (S_l - \not{p}_2 S_{l-1})\, u_1\,, \\ &\to S_k S_l (2p_\mu - \gamma_\mu) - S_k S_{l-1}[4p_\mu - (P+1)\,\gamma_\mu] \\ &\quad - S_{k-1} S_l \gamma_\mu + S_{k-1} S_{l-1}(2p_\mu - \gamma_\mu). \end{aligned}$$

The coefficients of γ_μ and of p_μ are

$$\gamma_\mu : -(S_k + S_{k-1})(S_l + S_{l-1}) + (P+2)\, S_k S_{l-1}\,, \tag{4.9b}$$

$$p_\mu : 2[S_k(S_l - S_{l-1}) - S_{l-1}(S_k - S_{k-1})]. \tag{4.10b}$$

$(m = 2k, n = 2l+1)$:

As with F_2, the coefficients of γ_μ and p_μ contributing to F_3 are

$$\gamma_\mu : -(S_k + S_{k-1})(S_l + S_{l-1}) + (P+2)\, S_{k-1} S_l\,, \tag{4.9c}$$

$$p_\mu : 2[-S_{k-1}(S_l - S_{l-1}) + S_l(S_k - S_{k-1})]. \tag{4.10c}$$

$(m = 2k+1, n = 2k+1)$:

$$\begin{aligned} F_4 &= \bar{u}_2(S_k - \not{p}_1 S_{k-1})\, \not{p}_1 \gamma_\mu \not{p}_2 (S_l - \not{p}_2 S_{l-1})\, u_1\,, \\ &\to S_k S_l [4p_\mu - (P+1)\,\gamma_\mu] - (S_k S_{l-1} + S_{k-1} S_l) \\ &\quad \times (2p_\mu - \gamma_\mu) + S_{k-1} S_{l-1} \gamma_\mu\,. \end{aligned}$$

The coefficients of γ_μ and of p_μ are

$$\gamma_\mu : (S_k + S_{k-1})(S_l + S_{l-1}) - (P+2)\, S_k S_l\,, \tag{4.9d}$$

$$p_\mu : 2[S_k(S_l - S_{l-1}) + S_l(S_k - S_{k-1})]. \tag{4.10d}$$

If we denote "the integral part of x" by $[x]$, the sets of formulae (4.9) and (4.10) have each a single expression in terms of m and n, defining the general element (4.2). In fact, we have

$$F = \bar{u}_2[U(P)\,\gamma_\mu - 2V(P)\,p_\mu]\,u_1\,, \tag{4.11}$$

where p_μ obeys Rule 1 and

$$U(P) = (-1)^{m+n} \left[(S_{[\frac{1}{2}m]} + S_{[\frac{1}{2}m]-1})(S_{[\frac{1}{2}n]} + S_{[\frac{1}{2}n]-1}) - (P+2)(S_{[\frac{1}{2}(m-1)]} S_{[\frac{1}{2}(n-1)]})\right], \tag{4.12}$$

and

$$V(P) = (-1)^m S_{[\frac{1}{2}(m-1)]}(S_{[\frac{1}{2}n]} - S_{[\frac{1}{2}n]-1}) + (-1)^n S_{[\frac{1}{2}(n-1)]}(S_{[\frac{1}{2}m]} - S_{[\frac{1}{2}m]-1}). \tag{4.13}$$

The argument of each S in (4.12) and (4.13) is P.

Since (4.5) holds for $k > 0$, (4.12), and (4.13) are valid for $m \geqslant 2$ and $n \geqslant 2$. However, (4.5) is true for $k = 0$ also provided that we define $S_{-1}(P) \equiv 0$. With this understanding, (4.12) and (4.13) are valid for $m \geqslant 0$ and $n \geqslant 0$, taking $[\frac{1}{2}(m-1)] = -1$ when $m = 0$.

Formulae (4.11), (4.12), and (4.13) together with Rule 1, define immediately the form factors arising from any matrix element (4.2). Equation (4.11) can be written

$$F = \bar{u}_2[\{U(P) - 2V(P)\}\gamma_\mu + iV(P)\,\sigma_{\mu\nu}q_\nu]\,u_1\,. \tag{4.14}$$

We can check formulae (4.12) by putting $P = 2$; (4.14) should then become Eqs. (3.2) and (3.3) of I. Since $S_j(2) = j + 1$, (4.13) gives

$$\begin{aligned} V(2) &= (-1)^m([\tfrac{1}{2}(m-1)] + 1) + (-1)^n([\tfrac{1}{2}(n-1)] + 1) \\ &= -g_{mn}\,, \end{aligned}$$

where g_{mn} is defined in *I*. It is also easy to show that

$$U(2) - 2V(2) = 1.$$

Thus (4.14) reduces correctly to (3.2) and (3.3) of *I* when $P = 2$.

It remains for us to identify m and n in an element of type (4.2) to which a general matrix element (4.1) reduces, using Rule 2. Let us consider the part

$$\gamma_\mu \Pi_b \not{p}_b u_1 \tag{4.15}$$

of (4.1). We label the $\not{p}_b$ as in Rule 1, so that a $+$ sign occurs immediately to the right of γ_μ, and define α and β as in Rule 1. Then $\Pi_b \not{p}_b$ reduces, *without* using (1.3), to one of the forms $E_1, \ldots, E_4$; each case needs separate consideration, and we detail the identification of n for E_2, as an example:

($\beta \leqslant 0, \alpha + \beta = 1$). (4.15) reduces to

$$\begin{aligned}\gamma_\mu E_2 u_1 &\equiv \gamma_\mu \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 \not{p}_1 u_1 \\ &= \gamma_\mu \not{p}_1 \not{p}_2 \cdots \not{p}_1 \not{p}_2 u_1 .\end{aligned}$$

Use of the Dirac equation has eliminated one $\not{p}_1$, so

$$n = 2(\alpha - 1) = 2 \mid \beta \mid ;$$

in (4.12) and (4.13)

$$[\tfrac{1}{2}n] = \alpha - 1 = \mid \beta \mid$$

and

$$[\tfrac{1}{2}(n - 1)] = \alpha - 2 = \mid \beta \mid - 1.$$

The values of n and the associated integers $\rho \equiv [\frac{1}{2}n]$ and $\sigma \equiv [\frac{1}{2}(n - 1)]$ can be expressed simply in terms of α or β in all four cases; these values are set out in Table I.

TABLE I

Values of n Associated with $\Pi_b \not{p}_b$

Condition on α, β	Reduced string (1.6)	n	$[\frac{1}{2}n] = \rho$	$[\frac{1}{2}(n - 1)\rangle = \sigma$
$\beta \leqslant 0$ $\alpha + \beta = 0$	E_1	$2\alpha = 2 \mid \beta \mid$	$\alpha = \mid \beta \mid$	$\alpha - 1 = \mid \beta \mid - 1$
$\beta \leqslant 0$ $\alpha + \beta = 1$	E_2	$2(\alpha - 1) = 2 \mid \beta \mid$	$\alpha - 1 = \mid \beta \mid$	$\alpha - 2 = \mid \beta \mid - 1$
$\beta > 0$ $\alpha + \beta = 1$	E_3	$2\beta - 1 = 2 \mid \alpha \mid + 1$	$\mid \alpha \mid = \beta - 1$	$\mid \alpha \mid = \beta - 1$
$\beta > 0$ $\alpha + \beta = 0$	E_4	$2\beta - 1 = 2 \mid \alpha \mid - 1$	$\mid \alpha \mid - 1 = \beta - 1$	$\mid \alpha \mid - 1 = \beta - 1$

The results of Table I can be expressed very simply,

$$\begin{aligned}&\text{If} \quad \beta \leqslant 0, n = 2 \mid \beta \mid \\ &\text{If} \quad \beta > 0, n = 2\beta - 1.\end{aligned} \tag{4.16}$$

We can derive a similar rule for the product $\Pi_a \not{p}_a$ in (4.1) by labelling the $\not{p}_a$ alternately $+$ and $-$, starting from the *right* with a $+$ sign.

We then define

$$\gamma = (\text{number of } \overset{+}{p}_2) - (\text{number of } \overset{-}{p}_2), \tag{4.17}$$

$$\delta = (\text{number of } \overset{+}{p}_1) - (\text{number of } \overset{-}{p}_1). \tag{4.18}$$

Then δ plays the same role in $\Pi_a \not{p}_a$ as β does in $\Pi_b \not{p}_b$, and the analogue of (4.16) is

$$\begin{aligned} &\text{If } \delta \leqslant 0,\ m = 2\,|\,\delta\,| \\ &\text{If } \delta > 0,\ m = 2\delta - 1. \end{aligned} \tag{4.19}$$

Collecting together the results (1.11), (4.18), (4.16), (4.19), (4.12) and (4.13), and referring to Table I, we obtain the algorithm for evaluating (4.1):

RULE 6.

(i) In a matrix element of the form

$$\bar{u}_2 \prod_a \not{p}_a \gamma_\mu \prod_b \not{p}_b u_1 ,$$

where each a or b takes the value 1 or 2, label the $\not{p}_a$ and the $\not{p}_b$ alternately with $+$ and $-$ signs, starting in each case with a $+$ sign on the $\not{p}$ adjacent to γ_μ.

(ii) Define, for the string $\Pi_b \not{p}_b$ to the right of γ_μ,

$$\beta = (\text{number of } \overset{+}{p}_2) - (\text{number of } \overset{-}{p}_2)$$

and, for the string $\Pi_a \not{p}_a$ to the left of γ_μ,

$$\delta = (\text{number of } \overset{+}{p}_1) - (\text{number of } \overset{-}{p}_1).$$

(iii) Define n, ρ and σ by

$$n = 2\,|\,\beta\,|, \quad \rho = |\,\beta\,| \quad \text{and} \quad \sigma = |\,\beta\,| - 1 \quad \text{when} \quad \beta \leqslant 0$$

$$n = 2\beta - 1 \quad \text{and} \quad \rho = \sigma = \beta - 1 \quad \text{when} \quad \beta > 0.$$

Likewise define m, μ, and ν by

$$m = 2\,|\,\delta\,|, \quad \mu = |\,\delta\,| \quad \text{and} \quad \nu = |\,\delta\,| - 1 \quad \text{when} \quad \delta \leqslant 0$$

$$m = 2\delta - 1 \quad \text{and} \quad \mu = \nu = \delta - 1 \quad \text{when} \quad \delta > 0.$$

(iv) Define

$$U(P) = (-1)^{m+n}[(S_\mu(P) + S_{\mu-1}(P))(S_\rho(P) + S_{\rho-1}(P)) - (P + 2)(S_\nu(P)\, S_\sigma(P))]$$

and

$$V(P) = (-1)^m S_\nu(P)[S_\rho(P) - S_{\rho-1}(P)]$$
$$+ (-1)^n S_\sigma(P)[S_\mu(P) - S_{\mu-1}(P)],$$

where $S_k(P)$ is a Chebyshev polynomial of the first kind for $k \geqslant 0$, and

$$S_{-1}(P) \equiv 0.$$

(v) The matrix element is then given by (4.11) or (4.14).

Rule 6 should reduce to Rule 5 of *I* when $P = 2$. Note first that the sign labelling in Rule 6 above is slightly different from that in Rule 5 of *I*: the signs in $\Pi_b\, \not{p}_b$ in (4.1) have all been changed, while those in $\Pi_a\, \not{p}_a$ remain the same. Remembering this, Rule 5 of *I* states that the contribution to $G(0)$ is

$$g_{mn} = -\beta - \delta, \tag{4.20}$$

where β and δ are defined in Rule 6. But (4.14) tells us that this should be equal to $V(2)$, which we have checked to be given by

$$V(2) = (-1)^m\, [\tfrac{1}{2}(m + 1)] + (-1)^n\, [\tfrac{1}{2}(n + 1)]. \tag{4.21}$$

So expressions (4.20) and (4.21) should be equal. The relation between β and n is given by (4.16), and it follows that

$$(-1)^n\, [\tfrac{1}{2}(n + 1)] = -\beta$$

for all β. Since δ and m are similarly related, we have checked that $g_{mn} = V(2)$.

5. Conclusion

Rules 1, 3, 4, 5, and 6 provide simple algorithms for calculating the contributions to the electromagnetic form factors from any allowable combination of γ-matrices. The contributions are given in terms of Chebyshev polynomials with argument $P = 2 - q^2$, where q is the photon 4-momentum.

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